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THE DEFINITION OF SIMILARITY

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I suggest for the teaching of elementary geometry a change in the customary definition of similar figures. The altered definition will not only be more generally applicable, more nearly commensurate with the idea which we all have of similarity, but will tend towards simplifying and unifying the teaching of certain separate parts of the subject of elementary geometry.

No change is proposed in the definition of similar triangles; leave that as it is. The change suggested is to define similar figures in general, as follows:

Two figures are similar when any three points in one form a triangle similar to the triangle formed by the three corresponding points in the other.

Probably every student of geometry even now gets an adequate idea of similarity, at least in plane figures; maps and plans are accepted at once as similar, corresponding lines are intuitively selected, nobody raises a whisper of objection when it is pointed out that the ratio of the areas of two similar figures, no matter how irregularly curved the boundaries, is the square of the ratio of two corresponding lines. But there is a gap in the argument. We have a definition of similar triangles, and of similar polygons; circles we prudently ignore, as we do all curvilinear figures; and with solids we are apparently helpless, for we confine ourselves to pointing out relations between ratios of certain pet lines, like radii, in figures of certain specified types, like spheres, or cylinders generated by the revolution of similar rectangles.[*] We may hope that the student concludes that there is a very important idea, applicable to all types of figures; but this idea we fail to define with any generality or to apply with scientific thoroughness.

It is true that some teachers make use of what they call perspective,[**] or homothetic[†] position, for defining similarity of plane figures; and that this may be extended to solid figures. This must seem to the high school pupil somewhat remote from the methods of constructing or of analyzing the figures concerned, as may be seen in trying to show that two maps of the State of New York may be placed homothetically, and that then points of the same name would be in homothetic correspondence. Two models of the same solid figure would be even more perplexing. The one luminous case of advantage in teaching is that of parallel sections of a pyramid, and of course we are not throwing that advantage away. Against the general use of such a definition and method, however, is the fact that similarity belongs to the figures themselves, and not to their relative position.

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[*] There is a cumbrous definition for similar polyhedra: See Phillips & Fisher, Elem. Geom. § 714

[**] Young & Jackson, Plane Geometry, p. 179.

[†] Hadamard, Geometrie Elementaire, chapitre V.

The definition of similar figures here proposed is not really new. It is substantially the same as saying that similar figures are distinguished from each other by a change of scale. As to the question of selecting corresponding points, it is best to consider first how we construct, by points, a figure similar to a given figure. Let the given figure be $A\,B\,C\,D$ . . . . . , the similar figure to which shall be named $A'\,B'\,C'\,D'$ . . . . . ; and the distance AB be x. Take any two points at a distance x', calling either A' and the other B'. Construct $A'\,B'\,C'$ on $A\,B\,C$, $A'\,B'\,D'$ *on* $A\,B\,D$, and so on. In these two figures any two corresponding lines (lines joining corresponding points) have the ratio $x' : x$.

If two figures are given similar, we must have some rule for selecting the first two pairs of corresponding points; and we shall need to know whether the rotational order in the two given figures is the same or not. Other corresponding points can now be picked out by constructing similar triangles. The following theorems are obvious and convenient.

Points that divide corresponding distances in the same ratio are corresponding points.

Lines that divide corresponding angles in the same ratio are corresponding lines.

Lines that pass through corresponding points and make equal positive angles with corresponding lines are corresponding lines.

The intersections of corresponding lines are corresponding points.

If, instead of having the similarity presupposed, we have two figures* whose similarity is in question, we shall need a rule for selecting corresponding points. Wherever such a need arises in elementary geometry, there is a rule of correspondence, and the rule is in general more complicated in statement than in application. Thus, in similar triangles, the angles correspond according to their size, and the sides and vertices as related to them; or the sides correspond according to relative length, and their intersections and angles as related to them. In circles we may select any point whatever on one circumference to correspond with any point whatever on the other, and succeeding points according to their interval in degrees from these two first points. The centres always correspond, and the rotational order of corresponding points is either the same or opposite, in the two circles, for all points. It is necessary that pupils should know how to select corresponding points, but no one, of course, would advocate the learning by heart of such rules as those here given. One learns these rules generally in the same way that one learns the rules for personal conduct or for equilibrium in walking. They are necessary, whatever definition of similarity is adopted.

The investigation of circles as similar figures requires a study of the triangles belonging to a circle. Such are central triangles (made by two radii and the chord joining their ends), inscribed triangles, circumscribed triangles, and others. For this study we develop the theorems about central angles, inscribed angles, angles of tangent and chord, tangent and secant, two chords, two secants, two tangents (circumscribed angle). Even without most of the theorems just referred to, we can prove that circles are similar figures, and that arcs of the same number of degrees are similar figures; but the theorems are very convenient; for example, in studying regular polygons.

From this point of view, the subject of angles connected with a circle, instead of being a digression as in the usual treatment, becomes contributory to the general idea of similarity.

It is an obvious consequence of our definition that "pairs of similar triangles, similarly put together, give similar figures; and every pair of similar figures (polygons) is composed of similar triangles similarly put together"*; and it is consequently proved that the ratio of the areas of similar polygons is the square of the ratio of similitude, as well as that the ratio of the perimeters is the same as the ratio of similitude. The analogous theorems about similar polyedra are directly obtained in the same way, that about volumes depending on the study of a tetraedron.

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* DeMorgan, quoted in Heath's Euclid, vol. II, p. 232.

When we come to similar curvilinear figures, the areas being determined as the limits of the areas of variable polygons, the proposed definition is of immediate advantage. For the variable polygons in the two figures are determined by corresponding points, are consequently similar, and have as ratio of areas the square of the constant ratio of similitude. Their limits therefore have the same ratio. There is no reason why this conclusion, even in elementary geometry, should be confined to circles; it must be remembered, however, that proving the number $\frac{S}{r^2}$ to be constant is a different thing from determining its value.

Let us suppose, for example, that we have two convex, irregularly curved, but similar, closed contours, the areas enclosed by which are to be compared. Within either contour draw as many square units (inches, say) as possible, contiguous and completely included by the contour. Then, in addition, draw, in the same way, as many squares as possible, one-tenth of an inch on a side, one hundredth of an inch in area; then the next smaller subdivision (one ten-thousandth of an inch in area), and so on. There will result a polygon less than the area enclosed by the contour. By annexing to this polygon, at any completed stage of its development, additional squares of the smallest size appearing in it, another polygon is derived which completely encloses the area of the figure.

If now any rectangle be drawn around the contour with its sides parallel to the sides of the enclosed squares, each square of the super-added set will have a place to itself on one side of the rectangle, reached by sliding on a line perpendicular thereto. The inner polygon, then, without the super-added squares, will differ from the area enclosed in the contour by less than the area of a complete row of these smallest squares arranged along the four sides of the rectangle. That is, the polygon differs from the area in question by less than $\frac{p}{10^n}$, where p is constant, and n may be (theoretically) increased as much as we like. The inner polygon, then, approaches the required area as its limit.

Within the other contour another inner polygon can be constructed, similar to that within the first contour, by locating for vertices the points corresponding to the vertices of the first inner polygon. The areas of the polygons will have for ratio the square of the ratio of similitude of the two given contours. Since the areas inclosed by the contours are respectively the limits of the areas of the inner polygons, the ratio of the areas inclosed by the contours is the square of the ratio of similitude.

Figures with concavities in their boundaries can of course be dissected into convex figures.

An obvious extension of this argument proves that similar solids, whatever their boundaries, have for ratio of volumes the cube of the ratio of similitude.

The rectangular figures just described have variable boundaries, but the limits of those boundaries are, of course, *not* the boundaries of the areas or volumes measured.

If now we define the length of a convex contour as the limit of the length of the perimeter of an inscribed convex polygon, the inscribed polygons in two similar contours having for vertices corresponding points; and if we define the area of a curved convex bounding surface as the limit of the area of an inscribed convex polyedron, we have complete command of the important numerical relations due to the similarity of any sort of figures, solid or plane. And this command does not depend upon hazily conceived analogies, or upon the authority of the learned teacher.

Those who have been reading the reports of the National Committee on Secondary Mathematics, especially that on the functional idea, will perhaps see in this definition a further advantage, in that it emphasizes correspondences of numbers and of geometrical entities.